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Quantum Relativity

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We propose a unified description of the known forces. We formulate a quantum relativistic spacetime as a (directed) graph of causal arrows with indefinite Hilbert metric, whose physical meaning is given. The simplest graph whose quantum relativity supports conservation of energy-momentum also supports a semidirect product of the cyclic groups **2** and **3** and the four-group $\mathbf{2}^2$. We call these lattice degrees of freedom (permutational) twain, trine, and spin. Quantized $\mathbf{2}^2$ becomes Lorentz *Spin*(4). Gauged, the energy-momentum and spin groups lead to gravity and torsion, and twain and trine lead to SU_2 and SU_3. We infer that color is actually trine, and the z component of isospin is twain.

In this paper we formulate a concept of quantum relativity and apply it to the simplest spacetime that supports energy-momentum conservation, based on $\mathbb{N}^4$, the quadruples of natural numbers. We describe how we quantize these structures (Section 1), the quantum structures that result (Section 2), and how these condense into just the right spacetime and standard-model invariance and gauge groups (Section 3).

1. QUANTUM RELATIVITY

From canonical quantization we abstract Dirac's quantum transformation theory, regarded as an extension of relativity. We define a noncommutative operator algebra for nonquantum theories also, similar to that of Sudarshan (1990), and regard every nonquantum theory as obtained from a quantum theory by restricting experiments to one frame. This restriction loses phase information. To quantize is to requantize: to

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restore the phases that are lost by this restriction and then to relativize the frame. This leaves the operator algebra projectively unchanged, since its phases have no meaning in the nonquantum theory, and relativizes the subalgebra of variables, which special and general relativity leave absolute.

This process extends both relativity, which leaves the frame algebra fixed, and canonical quantization, which applies only to mechanical systems. Therefore we call it quantum relativity. It applies to any algebraic structure, including graphs, semigroups, and groups, and so to any algebraic spacetime model.

1.1. Nonquantum Operators

An *arrow* $(b \leftarrow a)$ is a mapping $a \rightarrow b$ with one point domain a and one-point codomain and range b, so that $(b \leftarrow a)a = b$; or else it is the arrow 0. The arrow $b \leftarrow a$ represents an impulsive action that sends a system from state a to b. The arrow 0 represents the impossible. We multiply arrows thus:

$$(c \leftarrow b)(b \leftarrow a) = (c \leftarrow a), \quad (d \leftarrow c)(b \leftarrow a) = 0 = (d \leftarrow c)b \quad \text{when } c \neq b \tag{1}$$

We write $T \leftarrow S$ for the set of arrows from points of S to points of T. Here $S \leftarrow S$ is a semigroup, the arrow semigroup (category) of S.

By an *operator* of any nonquantum system we mean a formal linear combination of arrows on its state space. The arrow algebra A is the linear space of all such operators, and the operator product extends the arrow product linearly. We write the operator algebra on a set S as $A = (S \leftarrow S)''$, the double prime indicating a double dual. A is the universal covering algebra of the semigroup $S \leftarrow S$. We write $(S \Leftarrow S)$ for the subsemigroup of identity arrows $t \leftarrow s$ with $t = s \in S$. Identity arrows represent sharp selection acts. They are minimal Hermitian idempotents in the commutative algebra $C = (S \Leftarrow S)''$ called the *frame algebra*. Most minimal Hermitian idempotents in A are outside C, but the nonquantum theory does not recognize them as selection acts. The quantum theory forgets C and puts all such operators on the same footing.

We represent any complex function $\lambda(s)$ on S by the operator $\lambda = \sum_{s \in S} \lambda(s)(s \leftarrow s)C$. We represent classes by their characteristic functions and thus by operators in C. Thus C includes and defines the elementary class and predicate algebra of the system.

We represent any mapping $f: S \rightarrow S$ by the linear operator $f = \sum_S (f(s) \leftarrow s) \in A$, not necessarily in the frame algebra. None of this has anything to do with quantum theory, as Sudarshan (1990) pointed out. It describes nonquantum theory, too.

Furthermore, we associate a Grassmann creator s and its adjoint $s^\dagger$ with each state $s \in S$, and identify an arrow $(b \leftarrow a)$ with the Grassmann product $ba^\dagger$ of an annihilator $a^\dagger$ for the tail and a creator b for the head. To treat an arrow $ba^\dagger$ as a unity in all further multiplications we brace it as $\{ba^\dagger\}$. The associative law of multiplication stops at the brace, as in set theory. We extend the brace operator to a linear operator: $\{\alpha + \beta\} = \{\alpha\} + \{\beta\}$. Thus the directed graph of $\mathbb{N}^4$ is represented by the Grassmann product of braced arrows joining vertices to their children (next future vertices).

Quantum relativity preserves the operator algebra A, but relativizes the frame algebra C. We suppose that each maximal commutative subalgebra is the frame algebra for some maximal quantum experimenter. Quantum relativity is frame relativity. Since the elements of A represent acts while those of C represent states, and A is absolute while C is relative, quantum relativity is based on acts rather than states, in the same sense that special relativity is based on spacetime events rather than space points.

We may think of the points $s \in S$ as basis vectors in the space S'' of formal linear combinations of points of S, and arrows as linear operators on S''. The arrow functor is cogredient to its head and contragredient to its tail. Thus under the transformation of one point a by $a \to 2a$, arrows transform according to $(a \leftarrow b) \to 2(a \to b)$, $(b \leftarrow a) \to \frac{1}{2}(b \leftarrow a)$, and $(a \leftarrow a)$ is invariant. Therefore the nonquantum logic is insensitive to the phases and magnitudes of the states $s \in A$. As long as we stay in the nonquantum frame based on identity arrows $s \leftarrow s$, superpositions do not occur, we cannot know the magnitudes and phases of the basis vectors s, and their relations can only be known projectively (up to factors). Anticommutation of states s, s' is indistinguishable from commutation, for example.

1.2. Rephasing

Nonquantum logic is quantum logic with an absolute frame. When we relativize the absolute frame C and do experiments that cross frames, the phases and magnitudes of the matrix elements of operators in A matter physically and must be restored to agree with experiment. We call this process rephasing. Before we relativize the frame, rephasing leaves the theory nonquantum and unchanged in content. For example, canonical quantization rephases the products of the momentum-and-position harmonics $\exp i(\alpha x + \beta p)$, for all $\alpha, \beta \in \mathbb{R}$, which commute projectively before and after quantization. Similarly, we quantum relativize any algebraic structure by forming its arrow algebra, rephasing, and relativizing the frame. Rephasing is most of the work.

1.3. Quantum Group

The algebraic quantization we proposed earlier (Finkelstein, 1972) replaced a commutative coordinate algebra by a noncommutative algebra rather than relativizing it. This resulted in a concept of quantum semigroup ("q dynamics") defined by a coproduct on the noncommutative coordinate algebra. Quantum relativity leads to a new concept of quantum semigroup and quantum group with a greater invariance.

We now quantum relativize the concept of group to obtain a concept of quantum group, and the standard model groups to find corresponding quantum groups. We regard the random element of the group as a physical entity, the *operon* of the group. The group operon becomes a quantum when we quantize the group, just as the electron is the quantum, not its algebra.

The operator algebra $B = (G \leftarrow G)''$ for a nonquantum semigroup G with semigroup product $b \circ a$ has two natural products. The first product is that already defined for any operator algebra. We call it the series product to distinguish it from the second, which we call the parallel product. The parallel product of $(d \leftarrow c)$ and $(b \leftarrow a)$ is

$$(d \leftarrow c) \circ (b \leftarrow a) := (d \circ b \leftarrow c \circ a) \tag{2}$$

A linear space B with two unital associative algebra products, called series and parallel, we call a *double algebra*. Its elements we call *double operators*.

Consider first the operator algebra $B = (G \leftarrow G)''$ of a semigroup. B is a double algebra with a preferred frame $C = (G \Leftarrow G)''$ and a natural †-operation, arrow reversal combined with complex conjugation. The identity of the series product of B is the usual identity operator $I = \sum_{g \in G} (g \leftarrow g)$. The identity of the parallel product of B is $U = (u \leftarrow u)$, called the unit operator, where u is the identity group element. U is the projection of the group unit. The double operator representing the inverse $g \rightarrow g^{-1}$ is

$$\mathrm{Inv} = \sum_G (g^{-1} \leftarrow g) \in B \tag{3}$$

also called the antipode. For any $\alpha \in A$ we designate $\mathrm{Inv}\, \alpha\, \mathrm{Inv}$ by α^{Inv}. The inverse obeys

$$\mathrm{Inv}\,\mathrm{Inv} = I, \qquad \alpha^{\mathrm{Inv}} \circ \alpha = U = \alpha \circ \alpha^{\mathrm{Inv}}, \qquad (\beta\alpha)^{\mathrm{Inv}} = \alpha^{\mathrm{Inv}} \beta^{\mathrm{Inv}} \tag{4}$$

and

$$\mathrm{Inv} = \mathrm{Inv}^{\dagger} \tag{5}$$

The frame C is generated by the identity arrows $g \leftarrow g$. Because our product is not that of a Hopf bialgebra, these are not the usual defining properties of an antipode.

We designate the two exponential operations based on the series product $\beta\alpha$ and parallel product $\beta \circ \alpha$ by exp: $B \to B$ and Exp: $B \to B$, respectively.

Now we relativize the frame. A quantum group (actually, operon) is then defined by a double algebra B with an element $\mathrm{Inv} \in B$ obeying (2). B defines the operon in the same sense that an algebra of operators defines a quantum. Without Inv, a double algebra defines a quantum monoid. A quantum †-group is a quantum group with an adjoint †: $B \to B$ obeying (3).

We mention briefly the difference between this concept of q group and the previous one (Finkelstein, 1972). Let Δ: $G \otimes G \to G$ be the equalizer product on G, defined by $g\Delta h = 0$, $g \neq h$, $g\Delta g = g$. This product depends on the logical structure of G, not the algebraic. One may define a product on $B = (G \leftarrow G)''$ by

$$(d \leftarrow c) \circ' (b \leftarrow a) := (d \circ b \leftarrow C\Delta a) \tag{6}$$

This is the pointwise product of G-valued functions on G. It depends on the phases, and breaks frame invariance, but it was appropriate for the old quantization process. We no longer use this product.

While the arrow process transforms any nonquantum semigroup into a unique quantum monoid, and any nonquantum group into a unique quantum group, it alone never seems to give the quantum group of physical interest, which always seems to require rephasing.

1.4. Coherent Group

To form a classical Lie group from a quantum or nonquantum group G we generalize coherent states. We use *finite* generators of the double algebra B as *infinitesimal* generators of a classical Lie group, called a coherent group of G. Call Γ a *core* of the double algebra B when:

1. Γ is a linear subspace of B.
2. The least sub-double-algebra of B that includes Γ is B itself.
3. Γ is a double Lie algebra (closed under both the operator commutator $\beta\alpha - \alpha\beta$ and the group commutator $\beta \circ \alpha - \alpha \circ \beta$).
4. No proper subset of Γ obeys 1–3.

By a *coherent unitary group* cG of a $\dagger$-group (or monoid) G, quantum or not, with core Γ, we mean a Lie group of parallel exponentials $\mathrm{Exp}(\gamma - \gamma^\dagger)$ of elements $\gamma \in \Gamma \subset B$. Whatever G is, cG is a nonquantum group. In addition, it has a form factor.

Notice how this concept generalizes the coherent states of a harmonic oscillator. In the present theory, we take the nonquantum semigroup $\mathbb{N}$ of the natural numbers under addition, abstracted from the energy levels of the oscillator, and the level number $n \in \mathbb{N}$, as basic for the oscillator, not the position and momentum, which arise after quantization and condensation. The operator algebra is $B = (\mathbb{N} \leftarrow \mathbb{N})''$. As core we take the excitation operator $c: n \mapsto n+1$; its adjoint $c^\dagger$, rephased so that $c^\dagger c - cc^\dagger = \tau 1$; the unit operator 1; and linear combinations thereof. The preferred coordinate frame C of B is the number frame, the subalgebra generated by the number operator $n = cc^\dagger$. The parallel product on B is defined by $(dc\dagger) \circ (ba^\dagger) = (db \leftarrow a^\dagger c^\dagger)$. This is the Wick normal-order product. All this expresses a nonquantum theory. We quantize by relativizing the frame. The coherent unitary group then consists of the exponentials $\exp_0(\alpha c - \alpha^* c^\dagger + i\varphi)$ for all complex α and real φ. The usual coherent states result from applying this group of operators to the point $n = 0$.

A coherent group of a quantum monoid can acquire physical meaning if it describes a physical condensation. In nature, however, condensation is not such a unique and simple mathematical process, but a complex sequential physical one. Different degrees of freedom freeze at different critical temperatures as a structure cools, their critical temperatures are sharply defined only for large aggregations, and in fermionic quantum structures pairing may take place before condensation. Group theory can provide a list of possible sequences of successive group reduction, but only a good experiment or dynamical calculation can tell us what sequence of coherent structures will actually be seen when a complex aggregate cools.

The coherent group is a nonsingular structure with generalized Gaussian form factors. The continuum limit is the singular limit as the scale time or chronon $\tau \to 0$.

The whole procedure that goes from some discrete skeleton theory (here the monoid $\mathbb{N}^4$) to a quantum theory, then to its coherent states, and finally to the singular limit of these coherent states (here $\mathbb{R}^4$) as the form-factor range (here τ) approaches zero, we shall call the qcs (quantization-condensation-singularization) process. It represents a physical Bose–Einstein-type condensation followed by observation under limited resolution.

2. QUANTUM SPACETIME

We now apply quantum relativity to a spacetime network model based on a directed graph of causal arrows. We have hypothesized that the Minkowski vacuum is a singular limit ($\tau \to 0$) of a coherent condensed state of a relativistic quantum network (Finkelstein, 1988). The simplest graph whose quantum theory supports energy-momentum conservation is that of $\mathbb{N}^4$. We call this graph $\delta\mathbb{N}^4$ (rather than $\mathbb{N}^4$) because it consists of the directed edges or arrows of $\mathbb{N}^4$ (rather than the points). Its quantum relativized form is $q\delta\mathbb{N}^4$.

Here we show that defects in $q\delta\mathbb{N}^4$ support the spacetime Poincaré group and exactly the unitary gauge groups of the standard model. While we study the simplest possible model, the results hold for many causal network structures with the same overall $\mathbb{N}^4$ symmetry semigroup.

2.1. The Classical Lattice

The simplest discrete spacetime model that allows for conservation of energy-momentum (the observed translation group) after the qcs process is the classical semi-infinite hypercubical checkerboard lattice $\mathbb{N}^4 = \{n^\mu \mid n^\mu \in \mathbb{N}, \mu = 1, 2, 3, 4\}$. We define four children of any event $n \in \mathbb{N}^4$ by adding 1 to any coordinate n^μ. We interpret the four axes of $\mathbb{N}^4$ as four symmetrically disposed null displacements, as of four light pulses emitted simultaneously from the center of a regular tetrahedron and absorbed at its vertices. Then the Minkowskian metric assumes the symmetric null form

$$g_{\mu\nu} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{7}$$

Unlike the Minkowski form, which distinguishes one axis from the other three, this form is invariant under the group S_4 of all permutations of the four coordinate axes. All its basis vectors are physical, while three out of four Minkowski basis vectors are supraluminal.

2.2. The Quantum Lattice

We now quantum relativize the classical semigroup $\mathbb{N}^4$. We first form its arrow algebra A, generated by the arrows $(n' \leftarrow n)$, $n, n' \in \mathbb{N}^4$, and its frame algebra C generated by the identity arrows $(n \leftarrow n)$. Before quantizing $\mathbb{N}^4$ we introduce four generators ι_μ like the excitation operators of four linear harmonic oscillators, obeying, however, the Lorentz covariant commutation relations and subsidiary condition

$$\iota_\nu{}^\dagger \iota_\mu = \iota_\mu \iota_\nu{}^\dagger + \tau g_{\mu\nu}, \qquad \iota_\mu{}^\dagger [0] = 0 \tag{8}$$

where [0] is the origin $n = 0$ of $\mathbb{N}^4$. Here the indefinite metric $g_{\mu\nu}$ enters where a delta function $\delta_{\mu\nu}$ would in canonical commutation relations. This implies that the metric defined by † is indefinite. The parallel product $\beta \circ \alpha$ on A is the normally ordered product (annihilators to the right of creators). In the spacetime interpretation the parameter τ is a fundamental time, the chronon.

To quantize, we forget C. We allow any frame, including the usual position or momentum frames of the quantum harmonic oscillators.

2.3. The Quantum Group of the Quantum Lattice

Now we quantize the symmetry monoid of the spacetime. The symmetry transformations of the lattice $\mathbb{N}^4$ form the monoid $G = S_4 \ltimes \mathbb{N}^4$, the semidirect product of the symmetric group (permuting the four lattice axes) with the translation monoid $\mathbb{N}^4$. G decomposes into the semidirect product

$$G = \mathbf{2} \ltimes (\mathbf{3} \ltimes (\mathbf{2}^2 \ltimes \mathbb{N}^4)) \tag{9}$$

Here **2** and **3** are the groups of the integers modulo 2 and 3, respectively. **2** is the quotient of S_4 by the alternating group A_4 and acts trivially on $A_4 = \mathbf{3} \ltimes (\mathbf{2}^2 \ltimes \mathbb{N}^4)$. **3** is the quotient of A_4 by the four-group $\mathbf{2}^2 = \{1, X_1, X_2, X_3\}$ of four commuting square roots of 1 and acts trivially on $\mathbf{2}^2$. These groups are composed of classes of permutations of the axes 1234 as follows:

$\mathbf{2}^2$ = the axis permutations $X_1 = (14)(23)$, $X_2 = (24)(13)$, $X_3 = (34)(12)$ and the identity 1. These are all the symmetry rotations X of a tetrahedron that obey $X^2 = 1$. We call this network degree of freedom *spin*.

$\mathbf{3}$ = cosets of $\mathbf{2}^2$ by the discrete rotations Y of the tetrahedron of $\mathbf{2}^2$ by $\pm 120°$ or $0°$ around a line from the center to one vertex, obeying $Y^3 = 1$. We call this network degree of freedom *trine* ["a group of three; ... the aspect of two planets when 120° apart" (*American Heritage Dictionary*)].

$\mathbf{2} \leftrightarrow$ cosets of **3** by the axis permutation (12) and the identity 1. Here $1 \in \mathbf{2}$ is the class of proper transformations, with determinant $+1$; $-1 \in \mathbf{2}$ is the class of improper transformations with determinant -1, respectively. This network degree of freedom we call *twain*.

This semifactorization of the group S_4 of order $4! = 4 \times 3 \times 2$ into groups of order 4, 3, and 2 is the famous one that Galois used to prove the solvability of the general quartic by radicals. The general quintic or higher-degree equation is not solvable by radicals because no such semifactorization into commutative groups exists for higher dimensions.

Before quantizing spin $\mathbf{2}^2$ we rephase the three elements X_α from commuting square roots of $+1$ to anticommuting square roots of -1. We then represent them by the three Pauli matrices $i\sigma_\alpha$ ($\alpha = 1, 2, 3$). We represent the identity in $\mathbf{2}^2$ by $i1$. These four modes generate the double algebra $(\mathbf{2}^2 \leftarrow \mathbf{2}^2)''$, which is the algebra of linear transformations $M_2 \leftarrow M_2$ of the Pauli 2×2 spin-matrix algebra M_2. The elements of $(\mathbf{2}^2 \leftarrow \mathbf{2}^2)''$ may be called double spin operators. Now each Pauli matrix σ_α defines a state of the nonquantum $\mathbf{2}^2$ operon. The parallel product of the double algebra is induced by the operator product of M_2, by

$$(\sigma_\delta \leftarrow \sigma_\gamma) \circ (\sigma_\beta \leftarrow \sigma_\alpha) = (\sigma_\delta \sigma_\beta \leftarrow \sigma_\gamma \sigma_\alpha) \tag{10}$$

The inverse in this double algebra is the identity, the group being involutory:

$$\text{Inv} = I \tag{11}$$

We supply the double algebra of double spin operators with the adjoint $\dagger$ induced by the Pauli adjoint on M_2, $\sigma^\dagger = \varepsilon\sigma^{*T}\varepsilon$. This implies an indefinite metric.

We now quantize spin $\mathbf{2}^2$ by relativizing the frame. (11) means that q^2 is involutory.

2.4. The Lattice Condensate

Let us now form the coherent condensate of the quantum lattice $\mathrm{q}\mathbb{N}^4$ and its symmetry monoid $\mathrm{q}G = \mathrm{q}S_4 \ltimes \mathrm{q}\mathbb{N}^4$.

We have already made sure that the spacetime-energy-momentum translation group $\mathbb{R}^4 \times \mathbb{R}^4$ emerges as the singular limit of the coherent translation group $\{\exp_0(\lambda^\mu \iota_\mu - \text{H.c.}) \mid \lambda^\mu \in \mathbb{C},\ \mu = 1, 2, 3, 4\}$ of the quantum monoid $\mathrm{q}\mathbb{N}^4$.

Similarly, the Lorentz group now emerges as the classical limit of the quantum four-group $\mathrm{q}\mathbf{2}^2$. The coherent states $\exp_0(\vartheta^\alpha\sigma_\alpha - \vartheta^{\alpha *}{\sigma_\alpha}^\dagger)$ make up exactly the spin group $Spin_4$. Quantum spin is the outcome of the network spin degree of freedom.

That is, just as there is a qcs process that transforms $\mathbb{N}^4$ into $\mathbb{R}^4$, there is one that transforms $\mathbf{2}^2$ into the Lorentz spin group of $SO(1, 3)$ (with the choice of metric given). For this it is important that S_4 respects the symmetric null metric $g_{\mu\nu}$.

2.5. Unification of the Metrics

The indefinite spacetime metric of Minkowski is inextricably linked to an indefinite quantum (pseudo-Hilbert space) metric in this theory, even more so than in quantum electrodynamics. The spacetime metric is merely the quantum metric evaluated between two ψ vectors representing condensed coherent states. It seems necessary to give a consistent physical interpretation for the ψ vectors of norm $\psi^\dagger\psi = 0$ in such a metric.

We may found ordinary quantum mechanics on the following operational definitions of ψ vectors, their dual, φ, their contraction $\varphi : \psi$, and the adjoint $\dagger$:

1. Each nonzero ψ vector represents an input operation, and each nonzero dual vector represents an outtake operation.

2. If $N = (\psi^\dagger : \psi)(\varphi^\dagger : \varphi)$ is the number of trials of a transition $\varphi \leftarrow \psi$, then $T = (\psi^\dagger : \varphi)(\varphi^\dagger : \psi)$ is the expectation value of the number of transitions that actually occur.

The usual transition probability P is the ratio T/N. If $\psi^\dagger\psi = 0$, then $N = 0$. There is no contradiction; the transition is simply never tried. A quantum theory with indefinite metric includes some experiments with repetition rate 0.

This makes physical sense. For example, if we do a crossed-polarizer experiment with polarizers traveling at speed V down the optical bench relative to us, the repetition rate of the experiment approaches 0 as $V \to c$, and in the limit no photon crosses the first polarizer and the transition through the second is never tried. Thus it is not surprising that the quantum prohibition of transitions with null norm is connected with the speed limit of special relativity. An indefinite metric implies that besides the usual forbidden, allowed, and assured quantum transitions, there are some that we cannot even try.

We did not expect this synthesis of quantum and spacetime concepts. We started from the clash between unitarity, finiteness, and Lorentz invariance, there being no finite-dimensional unitary representations of *Spin*(4). On the basis of Einstein locality, we half expected that the metric $\dagger$, a seriously nonlocal concept, would become a dynamical variable in a more flexible quantum logic (Finkelstein, 1968). We expected another level of theory below the network topology, giving the dynamics of the network quantum metric, and could not imagine how it would look. We saw no physical meaning for an indefinite metric.

Thus we thought the new quantum metric would inherit definiteness from the usual quantum metric and variability from the spacetime metric. Instead it inherits constancy from the usual quantum metric and indefiniteness from the spacetime metric. The emergent gravitational spacetime metric varies merely because the network does, somewhat as the intrinsic metric of a surface imbedded in a larger Euclidean space varies because the subspace does. This means that there is a chance that we have already reached a fundamental level of theory. We now face a familiar dilemma: on the one hand the improbability of any one absolute dynamical law, and on the other the problem of formulating physics without one.

3. GAUGE FIELDS[2]

In addition to translation and spin degrees of freedom, particles have a triplet degree of freedom (color) on which strong SU_3 acts and a doublet one (weak isospin) on which weak SU_2 and U_1 act. These may be trine and twain. At the same time, the gravity gauge group derives from $\mathbb{N}^4$ and the torsional gauge group derives from spin $\mathbf{2}^2$ in S_4. This exhausts the symmetries of $\mathbb{N}^4$.

There are at least three ways in which a group G occurs in a space X:

As coordinate: We may describe a local object by a group element that transforms a specimen at the origin into the actual object. Here G is a coordinate space. There are additional invariant coordinates distinguishing the specimens at the origin.

As structural group: We may describe a string defect by the group element acting on a test cell that we carry around the string. G is then the structural group of a bundle.

As gauge group: We may go from invariance under G to invariance under the function group G^X of group elements varying with position in the lattice.

First we use G as a quantum space $X = G$ for a defect, which will then have the twain and trine degrees of freedom of S_4, as well as spin and translation. Then we use G as the basis for a gauge group. In a nonquantum theory, the gauge group based on a Lie group G is G^X, where X is the spacetime. Here we take $X = G$ and understand G^G as $(G \leftarrow G)''$, which is defined for quantum groups, too. In a principal bundle the fiber is the structural group; here the base is also. Then $\mathbf{2}$ becomes $\mathbf{2} \leftarrow \mathbf{2}$, $\mathbf{3}$ becomes $\mathbf{3} \leftarrow \mathbf{3}$, $\mathbf{2}^2$ becomes $\mathbf{2}^2 \leftarrow \mathbf{2}^2$, and $\mathbb{N}^4$ becomes $\mathbb{N}^4 \leftarrow \mathbb{N}^4$. Under qcs the arrow semigroup $\mathbf{n} \leftarrow \mathbf{n}$ becomes U_n (GL_n for complex parameters), $\mathbb{N}^4 \leftarrow \mathbb{N}^4$ becomes $\mathbb{R}^4 \leftarrow \mathbb{R}^4$, the familiar gauge group of gravity. Thus the factor $\mathbf{2} \leftarrow \mathbf{2}$ becomes GL_2 gauge and $\mathbf{3} \leftarrow \mathbf{3}$ becomes GL_3 gauge, whose real compact forms are U_2 and U_3, respectively. We identify these with weak U_2 and strong U_3. Thus twain is weak isospin and trine is color. Presumably $\mathbf{2}^2 \leftarrow \mathbf{2}^2$ becomes a torsional gauge.

We mention just two of many questions still unresolved. Is the U_1 in our U_2 weak hypercharge? If so, what is the U_1 in our U_3? Perhaps a dynamical theory can say.

[2]The results of Section 3 were found after the IQSA Meeting, where we presented only some of Sections 1 and 2.

4. SUMMARY

Since we do not have the dynamical action principle yet, there is not yet a unified theory of the known forces, but at least it is a unified description of them. To recapitulate: The simplest graph that supports energy-momentum conservation after the qcs process is $\delta\mathbb{N}^4$. Its unit cell is the discrete hypercube $\mathbf{2}^4$ of 16 points, partially ordered from (0, 0, 0, 0) to (1, 1, 1, 1) by increasing individual coordinates. The symmetry group of the directed graph $\delta\mathbf{2}^4$ is the symmetric group S_4 on its four axes. This is the famous semidirect product $S_4 = \mathbf{2} \ltimes (\mathbf{3} \ltimes \mathbf{2}^2)$ of the cyclic groups **2** and **3** and the four-group $\mathbf{2}^2 = \mathbf{2} \times \mathbf{2}$. We call the physical degrees of freedom corresponding to **2**, **3** and $\mathbf{2}^2$ twain, trine, and spin, respectively. Similarly, the symmetries of the network $\mathbb{N}^4$ consist of the permutations S_4 and the future translations $\mathbb{N}^4$. Coherent states of the quantum translation semigroup $\mathrm{q}\mathbb{N}^4$ form Minkowski spacetime, and coherent states of the quantum spin group $\mathrm{q}\mathbf{2}^2$ form the Lorentz spin group. This leaves twain and trine uninterpreted. Besides the translation and spin degrees of freedom, particles have a strong three-valuedness on which strong SU_3 acts and a weak two-valuedness on which weak SU_2 acts. We hypothezised that these are permutational trine and twain, just as the Lorentz spin two-valuedness is the spin of S_4 and energy-momentum is in $\mathbb{N}^4$. We then computed the gauge group of the defect space $G = S_4 \ltimes \mathbb{N}^4$ and found that it contains just the unitary groups U_1, U_2, and U_3 in addition to spacetime gauge groups leading to torsion and gravity. We interpret the groups U_2 of twain and U_3 of trine modulo their centers U_1 as the weak isospin SU_2 and strong color SU_3.

Some crucial elements of the theory will soon support or destroy this interpretation.

1. In general, the factor B in a semidirect product $B \ltimes A$ acts nontrivially on the factor A. In the gauge theory, this means that the gauge field for A is a source for that of B. In the present theory, **2** and **3** act trivially on the next group in the semidirect product $\mathbf{2} \ltimes (\mathbf{3} \ltimes (\mathbf{2}^2 \ltimes \mathbb{N}^4)$, while $\mathbf{2}^2$ acts nontrivially on $\mathbb{N}^4$. This must determine which of the gauge fields are sources for the others. The results must be consistent with the fact that all fields are sources of gravity, that the color gluons carry no weak isospin, that color and weak isospin are Lorentz scalars, and so forth. There are eight such determinations to check with known properties, and four determinations about torsion that are new. (One already expects torsion to couple to itself and gravity.) These straightforward determinations are under way.

2. Twain changes under parity while trine does not.

Many networks share the symmetries of $\mathbb{N}^4$. Physical particles are modulations in the high-frequency carried provided by the quantum network, and if the above tests are met, one way to tell which network occurs in nature is to compute the coupling constants and masses (and Weinberg angles) of such modulations and compare them with experience. That will also determine the size of the chronon τ. We cannot do this yet. In addition, the vacuum is possibly described not by a single network like quantum $\delta\mathbb{N}^4$, but by a superposition of many.

To put it simply, we conclude that spacetime is like a quantum relativistic Rubik cubic lattice with an indefinite Hilbert metric. This simile has not yet been used up. S. Golomb (1982; see also Rubik, 1987) semifactorized S_3 in the theory of the Rubik cubic lattice much as we semifactorize S_4 in our null hypercubical lattice. Both $\mathbb{N}^3$ and $\mathbb{N}^4$ have the suggestive symmetries **2** (twain) and **3** (trine); no other powers of $\mathbb{N}$ do. Gravity comes from lattice translations and torsion from spin. Color is trine, and the z component of weak isospin is twain.

A fuller account will appear elsewhere (Finkelstein, 1993).

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